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There are no abnormal solutions of the Bethe−Salpeter equation in the static model

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Abstract

The four-point Green's function of static QED, where a fermion and an antifermion are located at fixed space positions, is calculated in covariant gauges. The bound state spectrum does not display any abnormal state corresponding to excitations of the relative time. The equation that was established by Mugibayashi in this model and which has abnormal solutions does not coincide with the Bethe−Salpeter equation. Gauge transformation from the Coulomb gauge also confirms the absence of abnormal solutions in the Bethe−Salpeter equation.

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One of the puzzling features of the Bethe−Salpeter equation [1, 2, 3] is the appearance in the bound state spectrum of abnormal solutions corresponding to excitations of the relative energy (or relative time) [3, 4]. These states do not have nonrelativistic counterparts, neither do they appear in the hamiltonian formalism. The first example of such states was provided by the Wick−Cutkosky model [5, 6], where, in the one scalar photon exchange approximation, an infinite number of new states, with a new quantum number, appeared in the bound state spectrum for values of the coupling constant greater than a critical value. It was also established that these states did couple to the two-particle Green's function [7].

The fact that the presence of abnormal states is due to the excitations of the relative time was checked by the analysis of the problem in the static model [8], in which two fermions are held fixed at definite positions in space. In this case the whole set of excited states should come entirely from the relative time excitations. It was again found that in the one-photon exchange approximation an infinite number of abnormal states exist when the coupling constant exceeds a critical value.

It was suspected [5] that the appearance of abnormal solutions might be due to the one-photon exchange approximation and that multiphoton exchange contributions might remove them from the spectrum. In this connection the exact static model was analyzed by Mugibayashi [9]. Using hamiltonian formalism he derived a second order differential equation that he identified with the Bethe−Salpeter equation. This equation had again an infinite number of abnormal states (for massless photons); the situation was even worse than before, since these were now appearing for *any* value of the coupling constant.

The trouble with Mugibayashi's result is that it is in obvious contradiction with experimental data: no such states are observed in QED. [Mugibayashi's scalar interaction can be transposed to QED.] On the other hand, the one-photon approximation results might still escape experimental tests, since the QED coupling constant α is far below from the critical value at which abnormal states appear.

These troublesome results and conclusions have led us to reanalyse the static model. The key ingredient in our analysis is the fact that the Green's function of the static model is exactly calculable and hence an explicit check of its poles provides indication about the bound state spectrum. Only one pole, that of the (normal) ground state, is found. The contradiction that exists with Mugibayashi's result stems from the fact

that his equation is *not* the Bethe−Salpeter equation, but rather a secondary equation of the theory. Calculating the Bethe−Salpeter kernel from Feynman diagrams or from the inverse of the Green's function, it is found, contrary to Mugibayashi's observation, that the kernel contains, in covariant gauges, an infinite series of multiphoton exchange diagrams.

The above results are also verified in the Coulomb gauge, where the Bethe-Salpeter equation has only one bound state. Gauge invariance of the theory can then be used to provide additional justification to the absence of abnormal solutions of the Bethe-Salpeter equation in other gauges.

We therefore conclude that the Bethe−Salpeter equation does not have any abnormal solution in the exact static model. Since it is unlikely that excitations of the space coordinates induce by themeselves relative time excitations, one should expect that this result remains also true in the more general four-dimensional theory. Our result confirms Wick's conjecture [5] that abnormal states are spurious solutions due to the one-photon exchange approximation.

The absence of abnormal solutions in the static model has also its relevance for Heavy Quark Effective Theory, which is formulated as an expansion around the static model [10].

We now turn to the details of the calculations. We consider spinor electrodynamics in the static limit where one fermion and one antifermion are held fixed at definite space positions. The corresponding lagrangian density, in an arbitrary linear covariant gauge characterized by a parameter ξ, is:

$$\begin{aligned}\mathcal{L} = \overline{\psi}_1(i\gamma_0\partial^0 - m_1 - g_1\gamma_0 A^0)\psi_1 + \overline{\psi}_2(i\gamma_0\partial^0 - m_2 - g_2\gamma_0 A^0)\psi_2 \\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2(1-\xi)}(\partial_\mu A^\mu)^2 \ . \qquad (1)\end{aligned}$$

Since the spacelike components of the γ-matrices are absent, the fermion fields can be classified according to the eigenvalues of the matrices γ_0 ($+1$ for the fermion, -1 for the antifermion). In the Feynman gauge, the above lagrangian density becomes equivalent, for its mutual interaction part, to that of the scalar interaction considered in Ref. [9].

The four-point Green's function describing the scattering of the two particles is defined, up to a normalization factor, by the functional integral:

$$G(x_1,x_2,x_1',x_2') = \int D\psi D\overline{\psi} DA \psi_1(x_1)\psi_2(x_2')\overline{\psi}_2(x_2)\overline{\psi}_1(x_1') e^{iS(\psi,\overline{\psi},A)+i\int J_0 A^0}$$
$$\equiv < \psi_1(x_1)\psi_2(x_2')\overline{\psi}_2(x_2)\overline{\psi}_1(x_1') e^{iS(\psi,\overline{\psi},A)+i\int J_0 A^0} > \ , \qquad (2)$$

where $S = \int d^4x \mathcal{L}(x)$ is the action and J_0 is an external source for the field A^0. To calculate this integral we eliminate the interaction term in S by the following transformations of the fermion fields:

$$\psi_i(x) \to e^{-ig_i \frac{1}{\partial^0} A^0(x)} \psi_i(x) \ , \qquad \overline{\psi}_i(x) \to e^{ig_i \frac{1}{\partial^0} A^0(x)} \overline{\psi}_i(x) \qquad (i=1,2) \ , \qquad (3)$$

where $\frac{1}{\partial^0} A^0(x)$ designates a primitive of $A^0(x)$ with respect to x_0, $\frac{1}{\partial^0}$ being defined in momentum space as the principal value of i/k^0. The Jacobian of these transformations is unity and we obtain for G:

$$G = < \psi_1(x_1) e^{-ig_1 \frac{1}{\partial^0} A^0(x_1)} \psi_2(x_2') e^{-ig_2 \frac{1}{\partial^0} A^0(x_2')} \overline{\psi}_2(x_2) e^{ig_2 \frac{1}{\partial^0} A^0(x_2)} \overline{\psi}_1(x_1') e^{ig_1 \frac{1}{\partial^0} A^0(x_1')}$$
$$\times e^{iS_0(\psi,\overline{\psi},A)+i\int J_0 A^0} > \ , \qquad (4)$$

where S_0 is the free action .

The remaining integration in the functional integral is gaussian and yields in a straightforward way:

$$G \ = \ G_0 e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \ \frac{1}{k^2+i\epsilon} \ (g_{00} - \xi \frac{k_0^2}{k^2+i\epsilon}) \ \left| \widetilde{J}^0(k) + J^0(k) \right|^2} \ , \qquad (5)$$

where G_0 is the free two-particle propagator,

$$G_0 = G_{10} G_{20} \ ,$$
$$G_{i0}(x_i - x_i') = \theta(x_i^0 - x_i'^0) e^{-im_i(x_i^0 - x_i'^0)} \delta^3(\mathbf{x}_i - \mathbf{x}_i') \qquad (i=1,2) \ , \qquad (6)$$

and $\widetilde{J}^0$ is:

$$\widetilde{J}^0(k,x_1,x_2,x_1',x_2') \ = \ i\frac{1}{k_0}\Big[g_1(e^{ik.x_1} - e^{ik.x_1'}) - g_2(e^{ik.x_2} - e^{ik.x_2'}) \Big] \ . \qquad (7)$$

For $J^0 = 0$ one has:

$$G \ = \ G_0 e^{\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} (1 - \xi \frac{k_0^2}{k^2+i\epsilon}) \frac{\left| g_1(e^{ik.x_1} - e^{ik.x_1'}) - g_2(e^{ik.x_2} - e^{ik.x_2'}) \right|^2}{(k^2+i\epsilon)k_0^2}} \ . \qquad (8)$$

In the argument of this exponential the terms proportional to g_i^2 $(i=1,2)$ contribute to the mass and wave function renormalizations. After carrying out the latter renormalizations

(we shall continue denoting by the same notations the renormalized masses) and setting

$$g_1 = g_2 = g \ , \qquad \alpha \ = \ \frac{g^2}{4\pi} \ , \qquad \lambda \ = \ \frac{g^2}{4\pi^2} \ , \tag{9}$$

we obtain from Eq. (8) for the renormalized Green's function:

$$\begin{aligned} G = G_0 e^{i\Big[\chi_\xi(x_1-x_2)-\chi_\xi(x_1-x_2')-\chi_\xi(x_2-x_1')+\chi_\xi(x_1'-x_2')\Big]} \\ \times e^{i\Big[h_{1\xi}(x_1^0-x_1'^0)+h_{2\xi}(x_2^0-x_2'^0)\Big]} \ , \end{aligned} \tag{10}$$

where χ_ξ and $h_{i\xi}$ $(i = 1, 2)$ are defined as:

$$\begin{aligned} \chi_\xi(x) &= \lambda \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{i\mathbf{k}.\mathbf{x}}}{k^2 + i\epsilon} \, (1 - \xi\frac{k_0^2}{k^2 + i\epsilon}) \, \frac{(e^{-ik_0x^0} - 1)}{(-k_0 + i\epsilon)(k_0 + i\epsilon)} \\ &= -i\lambda \left\{ \frac{x^0}{2r} \ln\left(\frac{r + x^0 + i\epsilon}{r - x^0 + i\epsilon}\right) + \frac{(\xi + 2)}{4} \ln\left(\frac{-x^2 + i\epsilon}{r^2}\right) \right\} , \qquad r = |\mathbf{x}| \ , \end{aligned} \tag{11}$$

$$h_{i\xi}(x^0) = -i\lambda\frac{(\xi + 2)}{4} \ln(-m_i^2(x^{02} - r_0^2) + i\epsilon) \qquad (i = 1, 2) \ , \tag{12}$$

where r_0 ($\simeq 0$) is the renormalization short distance regulator. Notice that χ is an even function of x^0. The functions $h_{i\xi}$ come from the one-fermion propagator renormalizations. They represent the cloud of scalar and longitudinal photons usually appearing in covariant gauges [11]. Their effect in momentum space consists in replacing the pole of the propagator by a power law singularity. A similar effect is also contained in the second term of χ_ξ.

In the absence of radiative corrections ($h_\xi = 0$), formula (10) can also be obtained by directly summing the Feynman diagrams corresponding to the multiphoton excahanges and contributing to the fermion-antifermion scattering amplitude. Because of the linearity of the denominators of the fermion propagators in momentum space [Eqs. (6)], the summation technique of the eikonal approximation [12] can be applied in exact fashion, with the fermions kept off their mass shell. When radiative corrections are present, the calculations are more involved. The emission amplitude of n photons from a fermion line is completely determined, through successive uses of Ward-Takahashi identities, in terms of the fermion propagator. After this step, the eikonal summation technique can be applied with appropriate adaptations and formula (10) is reached.

In the following we shall use for the total and relative variables the notations:

$$\begin{aligned} X &= \frac{x_1+x_2}{2}\ , \quad X' \ = \ \frac{x'_1+x'_2}{2}\ , \quad t \ = \ X^0 - X'^0\ , \\ x &= x_1 - x_2\ , \quad x' \ = \ x'_1 - x'_2\ , \quad r \ = \ |\mathbf{x}_1 - \mathbf{x}_2| \ = \ |\mathbf{x}'_1 - \mathbf{x}'_2|\ , \\ P &= p_1 + p_2\ , \quad p \ = \ \frac{1}{2}(p_1 - p_2)\ . \end{aligned} \tag{13}$$

To extract from the Green's function (10) the bound state spectrum of the theory we let the time t tend to infinity, keeping x^0 and x'^0 fixed. Because of the cluster decomposition [2], the Green's function behaves in general for infinite time separation as:

$$G(x_1,x_2,x'_1,x'_2) \underset{t\to\infty}{\sim} \sum_n \phi_n(x_1,x_2)\overline{\phi}_n(x'_1,x'_2) \ = \ \sum_n \phi_n(x)e^{-iP_{0n}t}\ \overline{\phi}_n(x')\ . \tag{14}$$

In this limit the Green's function (10) yields only one exponential factor and behaves as:

$$G \underset{t\to\infty}{\sim} e^{-i(m_1+m_2-\alpha/r)t}\ . \tag{15}$$

This behavior signals the presence of one bound state with energy

$$P_0 \ = \ m_1 + m_2 - \frac{\alpha}{r}\ , \tag{16}$$

which is gauge invariant. Notice that the contributions of the scalar and longitudinal photon clouds in the functions χ_ξ and $h_{i\xi}$ have cancelled each other in the limit $t\to\infty$. This is a consequence of the electric neutrality of the bound state considered here.

The above result about the bound state spectrum can also be obtained by analyzing the Green's function in total momentum space. After the isolation of the ground state pole (16), no other infinite type singularities appear.

We conclude from this calculation that the static model has only one bound state, the normal one, and no abnormal states exist.

We next turn to the analysis of the problem by means of the Bethe-Salpeter equation. By functional methods [13] one establishes the following equation for the Green's function:

$$(i\gamma_{10}\partial_{x_{10}} - m_1)G(x_1,x_2,x'_1,x'_2) \ = \ i\delta^4(x_1-x'_1)G_{20}(x_2-x'_2) - ig\gamma_{10}\frac{\delta}{\delta J_0(x_1)}G(x_1,x_2,x'_1,x'_2)\ , \tag{17}$$

and a similar one with the interchanges $1\leftrightarrow 2$. Here G_{i0} $(i=1,2)$ is the free propagator of fermion i [Eq.(6)]. The complete expression of the Green's function [Eq. (5)] can then be used to transform these equations into integro-differential equations for G. Equivalently,

one can directly apply the operators G_{i0}^{-1} ($i=1,2$) on the expression (10) of G. One obtains the following two differential equations (we omit radiative corrections, which actually disappear from the subsequently derived wave equations):

$$(i\partial_{x10}-m_1)G(x_1,x_2,x_1',x_2') = i\delta^4(x_1-x_1')G_{20}(x_2-x_2') - \Big[\chi_\xi'(x_1-x_2)-\chi_\xi'(x_1-x_2')\Big] \times G(x_1,x_2,x_1',x_2')\ , \tag{18a}$$

$$(i\partial_{x20}-m_2)G(x_1,x_2,x_1',x_2') = i\delta^4(x_2-x_2')G_{10}(x_1-x_1') - \Big[\chi_\xi'(x_2-x_1)-\chi_\xi'(x_2-x_1')\Big] \times G(x_1,x_2,x_1',x_2')\ , \tag{18b}$$

where the prime on χ designates the derivative with respect to the temporal argument:

$$\chi_\xi'(x) \ \equiv\ \frac{\partial}{\partial x^0}\chi_\xi(x)\ . \tag{19}$$

[$\chi_\xi'(x)$ is an odd function of x^0.]

Continuing the procedure, one also obtains a second order differential equation for G:

$$(i\partial_{x10}-m_1)(i\partial_{x20}-m_2)G(x_1,x_2,x_1',x_2') \ = \ i^2\delta^4(x_1-x_1')\delta^4(x_2-x_2') + \Big\{ i\chi_\xi''(x_1-x_2) + \Big[\Big(\chi_\xi'(x_1-x_2)-\chi_\xi'(x_1-x_2')\Big)\Big(\chi_\xi'(x_2-x_1)-\chi_\xi'(x_2-x_1')\Big)\Big]\Big\} G(x_1,x_2,x_1',x_2')\ . \tag{20}$$

Wave equations satisfied by the Bethe$-$Salpeter wave function are obtained by formally taking in the previous equations for G the limit $t\to\infty$ [Eq. (13)], using the cluster decomposition (14) and eliminating by integration on X'^0 all the amplitudes ϕ_n but one [2]. One finds:

$$(i\partial_{10}-m_1)\phi(x_1,x_2) = -\Big[\chi_\xi'(x)+\frac{\lambda\pi}{2r}\Big]\ \phi(x_1,x_2)\ , \tag{21a}$$

$$(i\partial_{20}-m_2)\phi(x_1,x_2) = -\Big[-\chi_\xi'(x)+\frac{\lambda\pi}{2r}\Big]\ \phi(x_1,x_2)\ , \tag{21b}$$

$$(i\partial_{10}-m_1)(i\partial_{20}-m_2)\phi(x_1,x_2) = \Big[i\chi_\xi''(x) - \Big(\chi_\xi'^2(x)-(\frac{\lambda\pi}{2r})^2\Big)\Big]\ \phi(x_1,x_2)\ . \tag{22}$$

In the Feynman gauge these equations reduce to those obtained by Mugibayashi [9] with the hamiltonian formalism.

Equations (21a)-(21b) yield for ϕ a single bound state with the energy (16) and wave function

$$\phi \ = \ Ce^{-i(m_1-m_2)x^0/2}e^{-i(m_1+m_2-\alpha/r)X^0}\exp\Big(i\chi_\xi(x)\Big)\ , \tag{23}$$

with C, a constant.

Equation (22) is best analyzed in euclidean space for the relative time variable. Using Wick rotation [3, 5] and then setting $p_0 = ip_0^E$ or $x^0 = -i\tau$, Eq. (22) becomes (for simplicity we consider the equal mass case $m_1 = m_2 = m$):

$$-\left(\frac{P_0}{2} - m\right)^2\phi = -\frac{d^2}{d\tau^2}\phi - \frac{\lambda}{2(\tau^2+r^2)^2}\left[(2+\xi)\tau^2 + (2-\xi)r^2\right]\phi$$
$$-\left\{\left(\frac{\lambda\pi}{2r}\right)^2 - \lambda^2\left[\frac{1}{r}\arctan\left(\frac{\tau}{r}\right) - \frac{\xi\tau}{2(\tau^2+r^2)}\right]^2\right\}\phi\ . \qquad (24)$$

For large $|\tau|$ the potential behaves as $-(1+\xi/2)\lambda^2\pi/(2r|\tau|)$, indicating that for $\xi > -2$ there are an infinite number of normalizable solutions for any value of λ. These additional solutions were identified with the abnormal solutions of the Bethe$-$Salpeter equation. On the other hand, had we kept in the above equation only the linear term in λ (coming from the one-photon exchange contribution), we would have faced the same situation as in the Wick$-$Cutkosky model [5, 6] or its static analogue of Ref. [8]. In this case the additional solutions exist only for $\lambda > 1/(4(1+\xi/2))$.

It is clear that these abnormal solutions do not appear as poles of the Green's function because they do not satisfy the first order equations (21a)-(21b). At this stage the main objection one could raise is the fact that Eqs. (18a)-(18b) are not fundamental equations of Quantum Field Theory and are specific to the static model; they would not survive in more general cases and hence only Eq. (22) would remain as a bound state equation, leading to the existence of abnormal solutions. Leaving aside for the moment the feature that the very existence and the energies of the abnormal solutions are manifestly gauge dependent and in any event the latter could not appear as poles of Green's functions [14], Eq. (22) suffers from the following main drawback: in spite of appearances, it is *not* the Bethe$-$Salpeter equation. The reason for this is that the potential in Eq. (22) does not represent the Bethe$-$Salpeter kernel. If this was the case, then we would conclude that the series of multiphoton exchanges in the kernel stops at two photon exchanges with a corresponding local expression in x-space. The contribution of the two-photon exchange diagram to the kernel can be explicitly calculated. In the Feynman gauge it reads:

$$K_2(x_1, x_2, x_1', x_2') = \lambda^2\theta(x_1^0 - x_1'^0)\theta(x_2 - x_2'^0)\ \frac{e^{-im_1(x_1^0 - x_1'^0) - im_2(x_2^0 - x_2'^0)}}{[(x_1 - x_2')^2 - i\epsilon][(x_2 - x_1')^2 - i\epsilon]}$$
$$\times\ \delta^3(\mathbf{x}_1 - \mathbf{x}_1')\ \delta^3(\mathbf{x}_2 - \mathbf{x}_2')\ . \qquad (25)$$

This is not a local operator in the temporal variables and does not cope with the $O(\lambda^2)$ terms of the potential of Eq. (22).

More generally, the $O(\lambda^2)$ terms of Eq. (20), which are multiplicative functions and act on the four arguments of the Green's function, do not represent the explicit expression of the Bethe−Salpeter kernel, which should act on two arguments only. The latter can be calculated either from the Feynman diagrams or iteratively by identifying its global action with the multiplicative functions of Eq. (20). The general expression of the kernel cannot, however, be put in a compact form and will not be reported here. Its main feature is that the series of multiphoton exchange diagrams does not stop at a finite order. Because the Bethe−Salpeter kernel is directly related to the inverse of the Green's function [$K = G_0^{-1} - G^{-1}$], the bound state spectrum of admissible states of the Bethe−Salpeter equation should be the same as that of the Green's function itself.

The absence of abnormal states in the Bethe−Salpeter equation can also be verified using gauge invariance of the lagrangian density (1) (up to the gauge-fixing term) and working in the Coulomb gauge. In this gauge, the expression of the Green's function is obtained from Eqs. (10)-(11) by the replacement of the photon propagator contribution by $-1/\mathbf{k}^2$. This amounts to replacing h_ξ by zero and χ_ξ by χ_C with:

$$\chi_C(x) \; = \; -\frac{\alpha}{2r}|x^0| \; . \tag{26}$$

For large time separations, the Green's function behaves as:

$$G_C \underset{t\rightarrow\infty}{\sim} e^{-i(m_1+m_2-\alpha/r)t} \; ; \tag{27}$$

it displays one bound state with energy given by Eq. (16).

The equivalent equations to Eqs. (21a), (21b) and (22) become:

$$(i\partial_{10} - m_1)\phi_C(x_1,x_2) = -\frac{\alpha}{r}\theta(-x^0)\phi_C(x_1,x_2) \; , \tag{28a}$$

$$(i\partial_{20} - m_2)\phi_C(x_1,x_2) = -\frac{\alpha}{r}\theta(x^0)\phi_C(x_1,x_2) \; , \tag{28b}$$

$$(i\partial_{10} - m_1)(i\partial_{20} - m_2)\phi_C(x_1,x_2) \; = \; i\frac{\alpha}{r}\delta(x^0)\phi_C(x_1,x_2) \; . \tag{29}$$

The latter equation is actually the Bethe−Salpeter equation in the Coulomb gauge, because the instantaneity of the interaction in this gauge reduces the Bethe−Salpeter kernel to the one-photon exchange diagram. This equation has clearly one bound state solution,

independently of Eqs. (28a) and (28b), which confirms the fact the Bethe−Salpeter equation should by itself contain the full information about the bound state spectrum.

The gauge transformation operator between two covariant gauges, or a covariant gauge and the Coulomb gauge, for the Green's function is obtained by taking the ratio (in x-space) of their corresponding expressions; it reduces to the ratio of the exponential factors with the χ and h functions. Taking the large time separation one also deduces the gauge transformation between the corresponding Bethe−Salpeter wave functions. One finds:

$$\phi_{\xi'}(x_1,x_2) = e^{i[\chi_{\xi'}(x)-\chi_{\xi}(x)]}\phi_{\xi}(x_1,x_2)\ , \qquad \phi_C(x_1,x_2) = e^{i[\chi_C(x)-\chi_{\xi}(x)]}\phi_{\xi}(x_1,x_2)\ . \tag{30}$$

The above transformation laws, which reflect general properties of the Bethe-Salpeter wave function [15], imply that if a bound state exists in one gauge, it should also exist in any other gauge. This results from the fact that the exponential factors involve the difference of two χ's and hence do not qualitatively modify (in euclidean space) the dominant asymptotic behavior of the wave functions.

It can be checked that these transformations applied on Eqs. (21a)-(21b) yield the corresponding equations in the new gauge. However, the equivalence of the Bethe−Salpeter equations of two different gauges cannot be established in strong form. A general feature of the Bethe−Salpeter equation is that it is only weakly invariant under gauge transformations, that is that its (infinitesimal) variation is proportional to the starting wave equation. [This behavior can be established by writing the equation in the form $G^{-1}\phi = 0$ and then using the infinitesimal gauge transformations of G and ϕ.] In particular, using in Eq. (29) transformation (30) one obtains an equation that is weakly equivalent to the Bethe-Salpeter equation of the covariant gauge. Nevertheless, the general property of the invariance of the bound state spectra under gauge transformations [14], explicitly verified on transformations (30), ensures us that the latter equation has also one bound state.

The previous calculations can also be repeated with a massive photon with mass μ. For large time separations, the Green's function behaves as:

$$G \underset{t\to\infty}{\sim} e^{-i(m_1+m_2-\alpha e^{-\mu r}/r)t}\ . \tag{31}$$

We deduce that there is only one bound state, the normal one.

In summary, the exact expression of the four-point Green's function in the static model does not display any abnormal state. In the one-photon exchange approximation (in covariant gauges), the situation is similar to that of the Wick−Cutkosky model,

where abnormal solutions appear for sufficiently large values of the coupling constant. The present study shows that the multiphoton exchange contributions sum up to sweep away the previous phenomenon and to reestablish a normal structure of the bound state spectrum. Since the phenomenon of the abnormal states is solely due to the relative time excitations, the results obtained in the static model, where spacelike motion is frozen, should survive in the more general case of four-dimensional theory.

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